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Twisted condensates of quantised fields

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Abstract. We construct some quasi-free pure states of free quantised fields in $(1+1)$ dimensions that are localised in the sense of Knight. We consider massless or massive Dirac fields forming a $\mathrm{U}(n)$, $n \geq 1$, multiplet and subject it to a local gauge transformation. We also subject a doublet of massive Klein-Gordon fields to local SO(2) transformations. We find the conditions that the resulting automorphisms are spatial in Fock space. In some cases the conditions turn out to require that certain parameters, identified as the winding numbers of the gauge, are integers. It is argued that this integer labels states of various charge.

1. Introduction

The only known relativistic quantised fields obeying the Wightman axioms in $(3+1)$ dimensions are free and generalised free fields and their Wick powers. But these 'bare' fields give poor descriptions of realistic particles; for example, we expect charged particles to be accompanied by their clouds of soft photons, or gluons in QCD. Haag *et al* [24] have suggested that charged states ought to be obtained from the vacuum by a local automorphism of the observables. Of all the states to which this gives rise, the 'states of interest' will be those belonging to a relativistically covariant representation with positive energy.

When these ideas are applied to the free Dirac field of zero mass in $(1+1)$ dimensions, we have been able to find some non-Fock relativistic representations [2-3]. These might do to describe particles with a cloud around them. But the results of [4] prove that if the mass is positive, or in more than one space dimension, there are no quasi-free non-Fock covariant irreducible representations of the free field. We must therefore, in these cases, look for more realistic models of particles among the Fock states, which are, of course, covariant and of positive energy.

In this paper we show that Fock space contains some states obtained from the vacuum by a local gauge transformation, and which are labelled by topological quantum numbers. Moreover, they are localised in the sense of Knight [5]: this means that on a suitably chosen algebra of observables, the states coincide with the vacuum outside a compact set.

Let us recall a model where 'topological' quantum numbers arise, which can be identified with charge. Consider the Dirac field in $(1+1)$ dimensions; in its second quantised form, the theory is defined by the CAR algebra $\mathscr{A}$ generated from the smeared creation and annihilation parts, $\psi^+(f)$, $\psi^-(g)$, where f and g lie in the one-particle space $K = L^2(\mathbb{R}, \mathbb{C}^2)$. We identify K with the set of complex initial data for the solutions to the c-number Dirac equation $(\partial\!\!\!/ + m)\psi = 0$. Then the second quantised smeared Dirac field is $\psi(f) = \psi^+(f) + \psi^-(f)$ interpreted as

$$\psi(f) = \int_{\mathbb{R}} \psi(x) f(x)\,\mathrm{d}x.$$

The usual complex structure on K is not the 'physical' one [6]; the generator of time-evolution is not bounded below in K. We choose a different complex structure which has the same real part but a different imaginary part. The Poincaré group $\mathrm{P}_+^\uparrow$ acts on K, since it acts on solutions of the Dirac equation. This action preserves the real part of the usual scalar product on K and so defines automorphisms $\{\sigma_{a,\Lambda}, (a, \Lambda) \in \mathrm{P}_+^\uparrow\}$ of the CAR algebra $\mathscr{A}$ over K. These automorphisms are spatial in the 'physical' representation π_m of $\mathscr{A}$ [6], that is, are given by unitary operators $U(a, \Lambda)$:

$$\sigma_{a,\Lambda}(A) = U(a, \Lambda) A U^{-1}(a, \Lambda)$$

for all $A \in \pi_m(\mathscr{A})$, $(a, \Lambda) \in \mathrm{P}_+^\uparrow$. The representation π_m is determined by the choice of 'physical' complex structure on K, and this in turn is determined by the projection operator P_+^m onto the positive-energy states, defined by

$$P_\pm^m = \frac{1}{2}\left(1 \pm \frac{p}{\omega_m(p)}\gamma_5 \pm \frac{m}{\omega_m(p)}\gamma_0\right), \qquad m \geq 0, \tag{1}$$

where $\gamma_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\gamma_5 = \gamma_0\gamma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and p and $\omega_m(p) = (p^2 + m^2)^{1/2}$, act on the Fourier transform of K as multiplication operators. We shall use this notation for the operators $-\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ and $(-\mathrm{d}^2/\mathrm{d}x^2 + m^2)^{1/2}$ on $L^2(\mathbb{R}, \mathbb{C}^2)$. Then the physical representation is

$$\psi(g) = \pi_m(a(g)) = a_{\mathrm{F}}^*(\overline{P_-^m g}) + a_{\mathrm{F}}(P_+^m g). \tag{2}$$

Here, $g \in K$, a_{F} is the Fock representation using the usual complex structure of K, and the bar is complex conjugation in K. Equation (1) makes clear the dependence on m.

In the model discussed in [3, 7–8], we put $m = 0$ and we subject K to unitary multiplication operators $T_{\alpha\beta}$

$$(T_{\alpha\beta} f)(x) = \{\exp \mathrm{i}[\alpha(x) + \gamma_5\beta(x)]\} f(x). \tag{3}$$

In (3), α and β are C^∞ functions, $\mathbb{R} \to \mathbb{R}$ with $\alpha' \in \mathscr{D}$, $\beta' \in \mathscr{D}$ and $\alpha(-\infty) = \beta(-\infty) = 0$. It is known that for each α, β there is a unique automorphism $\tau_{\alpha\beta}$ of $\mathscr{A}$ such that

$$\tau_{\alpha\beta}(a(f)) = a(T_{\alpha\beta} f), \qquad f \in K. \tag{4}$$

We say that $\tau_{\alpha\beta}$ is induced by $T_{\alpha\beta}$. The equations of motion are invariant under the corresponding 'rigid' transformations

$$T^{\mathrm{R}}(\theta) f = \mathrm{e}^{\mathrm{i}\theta} f, \qquad T_5^{\mathrm{R}}(\theta_5) f = \mathrm{e}^{\mathrm{i}\theta_5\gamma_5} f, \tag{5}$$

where θ, θ_5 are now constants. The equations of motion are not invariant under (3).

The local gauge and axial gauge transformations are not always spatial. We prove [3, 8] that $\tau_{\alpha\beta}$ is spatial in π_0 if and only if α, β obey

$$\alpha(\infty) + \beta(\infty) = 2\pi k \qquad \alpha(\infty) - \beta(\infty) = 2\pi n \tag{6}$$

for some integers k, n.

The integers k, n are the number of twists in the top (bottom) component of the gauge field, respectively, and are related to the charge and axial charge created by the unitary operators implementing the $\tau_{\alpha\beta}$ [8]. According to the analysis of Labonté [9], k, n will be the Fredholm indices of $P_+T_{\alpha,1}P_+$ and $P_+T_{1,\beta}P_+$ respectively. These indices have been computed by Carey, Hurst and O'Brien, using the theory of Toeplitz operators [7] and indeed do turn out to be the number of twists, k and n: k is the net number of right-moving particles and $-n$ the net number of left-moving particles, antiparticles being counted negatively.

This very satisfactory 'topological' theory of charge might suggest that the requirement that gauge transformations be spatial is in itself a good physical principle. But, if in equation (6) k and n are not integers, then we get non-Fock representations, but ones that are $P_+^\uparrow$ covariant [3]. Such representations cannot be excluded on physical grounds. We must conclude that the C^* algebra programme does not predict charge quantisation for this model.

For free fields, the existence of such non-Fock covariant representations is an infrared phenomenon peculiar to $(1+1)$ dimensions [4]. There are spacetime covariant infrared representations of massless fields in $(3+1)$ dimensions [10] but these are not $L_+^\uparrow$ covariant. It is to obtain covariant states, then, that we investigate spatial automorphisms; the idea is to find localised states in Fock space carrying quantum numbers defined geometrically or topologically. We get the state $U_\tau\Psi_F$ by applying the unitary operator U_τ, that implements the automorphism τ, to the vacuum state Ψ_F.

So far we have discussed the field algebra $\mathscr{A}$. It is open to us to specify what the observable algebra is. Naturally we would want it to obey the Haag–Kastler axioms, and it is natural to take it to be the subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ of elements invariant under the exact internal symmetries, G_0 say, of the model. It can then be hoped that, while our quantum numbers specify states in the same representation of $\mathscr{A}$ (the Fock representation), they will indeed label inequivalent representations of $\mathscr{A}_0$.

This would certainly be true if the scheme suggested by Haag *et al* [24] applied. This is not so clear, though, as the scheme depends on the initial choice of the representation of the algebra of the observables on the vacuum sector and there is no clear-cut choice for this representation, as far as we can see.

For a Dirac field of any mass, rigid gauge transformations, i.e. $\psi'(x) = e^{i\theta}\psi(x)$ with θ independent of x, are always symmetries. One should then take $\mathscr{A}_0$ to consist of gauge invariant operators, and be interested in automorphisms mapping $\mathscr{A}_0$ to itself. In the real Clifford algebra theory we regard K as a real Hilbert space, with scalar produce $\mathrm{Re}\langle , \rangle_K$. An automorphism is then any orthogonal, real linear transformation T of K. However, to commute with rigid gauge transformations, such an orthogonal map T must obey $e^{i\theta}T = T\,e^{i\theta}$, i.e. T must be complex linear, i.e. unitary on K regarded as a complex Hilbert space. So we limit our discussion to unitary one-particle transformations.

2. The models

In this section we give the models and summarise the results as theorems. The proofs are deferred to § 3.

Model 1. We take a single Dirac field ψ of mass $m > 0$. Carey *et al* [7] show that in the massless case, of automorphisms of the form

$$\psi'(x)=\lambda(x)\psi(x) \qquad \lambda:\mathbb{R}\to M_2(\mathbb{C}) \tag{7}$$

only those where λ is diagonal can be implemented in the relativistic Fock representation. They also give an argument, based on continuity in mass, which shows that the same should hold in the massive case. However, the projection P_+^m is not norm continuous at $m=0$, so some readers may need an independent argument. Indeed, their result does follow from the methods of this paper. We therefore choose λ in (7) to be diagonal

$$\lambda(x)=T_{\alpha\beta}(x)=\exp\mathrm{i}(\alpha(x)+\beta(x)\gamma_5). \tag{8}$$

Again we choose $\alpha(-\infty)=\beta(-\infty)=0$ and take $q_5=\lim_{x\to\infty}\alpha(x)$,

$$q=\lim_{x\to\infty}\beta(x), \qquad \alpha,\beta\in C^\infty(\mathbb{R}) \qquad \text{with } \alpha',\beta'\in\mathscr{D}(\mathbb{R}).$$

Note that the transformations (7) are done on the field ψ at time $t=0$. The fields at a later time t are determined by the fields at $t=0$ by the equation of motion $(\partial\!\!\!/+m)\psi(x,t)=0$, and the action of the automorphism $\tau_{\alpha\beta}$ induced by $T_{\alpha\beta}$ on $\psi(x,t)$ is thus determined by linearity. The action on $\psi^*(x)$ is fixed since $\tau_{\alpha\beta}$ is to be a $*$-automorphism, and naturally its action on products of ψ and ψ^* is then uniquely determined. The action on $\psi(x,t)$ is not purely a local gauge transformation (unlike the massless case [3], a circumstance allowing the exact solution of the Schwinger model [8, 11]).

The 'exact symmetries' of the model can be expressed in terms of the 'rigid' automorphisms $\tau_{\mathrm{R}}(\theta)$, $\tau_{\mathrm{R}}^5(\theta_5)$ induced by the one-particle operators T_{R}, T_{R}^5

$$(T_{\mathrm{R}}(\theta)g)(x)=\mathrm{e}^{\mathrm{i}\theta}g(x) \qquad (T_{\mathrm{R}}^5(\theta_5)g)(x)=\exp(\mathrm{i}\theta_5\gamma_5)g(x), \tag{9}$$

θ, θ_5 being constants. These are symmetries of the theory for any value of θ, but only if $\theta_5=n\pi$: axial symmetry is broken by the mass.

We then have the following result.

Proposition 1. The automorphism $\tau_{\alpha\beta}$ is implemented in π_m if and only if $q=n\pi$, $n\in\mathbb{Z}$.

Remark 1. We observe that $\tau_{\alpha\beta}$ is implementable if and only if $T_{\alpha\beta}(\infty)$ is one of the implementable rigid transformations, i.e. one of the true symmetries making up the group G_0. Defining the observables $\mathscr{A}_0$ to consist of G_0 invariant elements, we see that the state we get from the vacuum, by applying the implementable $\tau_{\alpha\beta}$, is localised in the sense of Knight, and on $\mathscr{A}_0$, $\tau_{\alpha\beta}$ is the identity outside a compact set. In fact, our proofs work equally well when θ is not constant outside a compact set, but converges rapidly enough to a constant at $\pm\infty$.

Remark 2. A recent paper by Carey and Ruijsenaars [25] contains a result which can be combined with our proposition 4 (see below) to give stability for the charged states created by axial gauge transformations. For, if $\theta\in C^\infty(\mathbb{R})$ is such that $\theta(-\infty)=0$ and $\theta(\infty)=2\pi$, our proposition 4 implies that

$$\mathrm{Ind}\begin{vmatrix}\mathrm{e}^{-\mathrm{i}\theta/2} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\theta/2}\end{vmatrix}=0,$$

while their results on the Fredholm index imply that

$$\operatorname{Ind}\begin{vmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

From the additivity of the index we then get that

$$\operatorname{Ind}\begin{vmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{vmatrix} = \operatorname{Ind}(e^{i\gamma_5\theta/2}) = 1.$$

On the other hand, they take for granted the implementability of local gauge transformations such as $\left|\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{smallmatrix}\right|$, where $e^{i\theta}$ is their standard kink (called σ in their paper); such implementability follows at once from the proof of our proposition 1.

Model 2. Now consider two Dirac fields with any masses, m_1, m_2, positive or null, and the CAR algebra over the one-particle space $L^2(\mathbb{R}, \mathbb{C}^4) = L^2(\mathbb{R}, \mathbb{C}^2) \oplus L^2(\mathbb{R}, \mathbb{C}^2)$, the space of initial data for two Dirac equations in $(1+1)$ dimensions with masses m_1, m_2 respectively. The physical representation π_{m_1,m_2} is then the skew tensor product of π_{m_1} and π_{m_2}: explicitly

$$\pi_{m_1,m_2}(a(g)) = a_F^*(\overline{(P_-^{m_1} \oplus P_-^{m_2})g}) + a_F((P_+^{m_1} \oplus P_+^{m_2})g)$$

where $g \in L^2(\mathbb{R}, \mathbb{C}^4)$, a_F is the Fock representation (over $L^2(\mathbb{R}, \mathbb{C}^4)$ now), and $P_\pm^m$ are as in (1).

We subject the algebra to local automorphisms τ_θ defined by unitary operators T_θ on $L^2(\mathbb{R}, \mathbb{C}^4)$ of the form

$$(T_\theta g)(x) = \begin{pmatrix} \cos\theta(x) & 0 & \sin\theta(x) & 0 \\ 0 & \cos\theta(x) & 0 & \sin\theta(x) \\ -\sin\theta(x) & 0 & \cos\theta(x) & 0 \\ 0 & -\sin\theta(x) & 0 & \cos\theta(x) \end{pmatrix} \begin{pmatrix} g_{11}(x) \\ g_{12}(x) \\ g_{21}(x) \\ g_{22}(x) \end{pmatrix}, \tag{10}$$

θ being a C^∞ function which vanishes at $-\infty$ and whose derivative has compact support. Put $Q = \theta(+\infty)$. The corresponding rigid transformations, denoted $T^R(\rho)$, are those given by (10) when $\theta(x) = \rho$, independent of x. If $m_1 = m_2$ then the rigid automorphism $\tau^R(\rho)$ is spatial for all ρ; if $m_1 \neq m_2$, only the discrete symmetry $\rho = n\pi$, i.e. $\psi \to -\psi$, is left; n is to denote an integer.

Proposition 2.

(a) If $m_1 \neq m_2$ and both m_1 and m_2 are positive, then τ_θ is spatial if and only if $Q = n\pi$.

(b) If $m_1 = m_2 \neq 0$, τ_θ is spatial for all Q.

(c) If either m_1 or m_2 vanishes, then τ_θ is spatial if and only if $Q = 2n\pi$.

Model 3. This is the generalisation to the group $U(n) \times U(n)$ of models 1 and 2. For simplicity we only fully consider the cases where all the masses are positive and different, or they are all zero.

Thus our transformations are of the form

$$\psi_j'(x, 0) = \begin{pmatrix} U_{jk}(x) & 0 \\ 0 & V_{jk}(x) \end{pmatrix} \psi_k(x, 0) = W_{jk}(x)\psi_k(x, 0) = (T_{U,V}\psi(x, 0))_j \tag{11}$$

with $U(x)$, $V(x) \in \mathrm{U}(n)$, acting on n Dirac fields $\psi_1, \ldots, \psi_n$ of masses $m_1, \ldots, m_n$. We avoid off-diagonal elements since they are never implementable. As usual, $U(-\infty) = V(-\infty) = 1_n$ and U, V are C^∞ functions, constant outside a compact set. Let τ_W denote the induced automorphism.

The subgroup $\{W\colon U = V\}$ gives gauge transformations, and the subgroup $\{W\colon U = V^*\}$ gives axial gauge transformations. The corresponding rigid transformations are implementable if and only if they are symmetries of the theory.

Proposition 3.
(a) If $m_1 = \ldots = m_n = 0$, then τ_W is spatial if and only if $W(\infty) = 1_{2n}$.
(b) If all masses are non-zero, then τ_W is implementable if and only if $W(\infty)$ is a rigid symmetry of the theory. In particular, if all masses are unequal, $W(\infty) = \pm 1_{2n}$ gives the only spatial automorphisms.

Both in proposition 2 and proposition 3 we meet the curious phenomenon that by breaking the gauge symmetry, putting the masses unequal, we quantise the axial charge and seem to stabilise these states. Axial gauge symmetry is broken whenever the masses are positive. This quantises charge.

The gauge solitons, obtained when $U = V$, are not 'topologically stable', because we can prove the following.

Proposition 4. Let $U = V$ and $m_j > 0$, $j = 1, \ldots, n$. Then the index of $P_+ W P_+$ is zero, where

$$P_+ = \bigoplus_{j=1}^{n} P_+^{m_j}.$$

Although there is no associated quantum number absolutely conserved in time when $U = V$, the number of twists will give a broken quantum number labelling resonances.

Proposition 5. Let $m_1 = m_2 = \ldots = m_n = 0$. Then the net number of right-going particles is the index of $P_+ T_{U1} P_+$, and this is minus the winding number of det U about 0. The net number of left-going particles is the index of $P_+ T_{1V} P_+$, and this is the winding number of det V about 0.

This proposition is the obvious generalisation of the result of Carey *et al* [7] for $n = 1$ in the massless case.

Now we come to our boson model [12].

Model 4. We have two free massive scalar fields, ϕ_1 and ϕ_2 of masses $m_1 > 0$ and $m_2 > 0$, and consider transformations $\phi \to \phi'$ given by

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix}(x, 0) = \Theta(x) \begin{pmatrix} \phi_1(x, 0) \\ \phi_2(x, 0) \end{pmatrix}, \qquad \begin{pmatrix} \dot\phi_1'(x, 0) \\ \dot\phi_2'(x, 0) \end{pmatrix} = \Theta(x) \begin{pmatrix} \dot\phi_1(x, 0) \\ \dot\phi_2(x, 0) \end{pmatrix}, \qquad x \in \mathbb{R} \tag{12}$$

done at $t = 0$. Since bosons obey a second-order wave equation, the automorphism is specified only when the transformation of both ϕ and $\dot\phi$ is given. We choose $\Theta(x)$ of the form

$$\Theta(x) = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

$\theta(-\infty)=0$, $\theta(+\infty)=Q$, $\theta' \in \mathscr{D}(\mathbb{R})$. Actually, (12) is shorthand for a symplectic transformation on the one-particle boson space $L^2(\mathbb{R}, \mathbb{C}^2)$, and we chose it orthogonal to preserve the reality of ϕ_j and $\dot{\phi}_j$.

Because of the infrared singularity, (12) makes no sense when m_1 or m_2 is zero. Let τ_θ denote the induced automorphism of the CCR algebra of the two fields ϕ_1, ϕ_2, represented in the relativistic Fock space.

Proposition 6.

(a) If $m_1 = m_2$, τ_θ is spatial for any value of Q.

(b) If $m_1 \neq m_2$, τ_θ is spatial if and only if $Q = n\pi$.

Remark. The implementability is decided by the Shale criterion [13] for bosons. This is a quadratic condition, and we have no index theory for such operators. Nor is it clear whether the quantum number n obtained in (6b) is stable, and what it is in terms of the (many) conserved quantities of free fields.

We end this section with some comments on the models. When these local automorphisms correspond to a rigid symmetry, and there are no twists, then we usually get a one-parameter group of implemented gauge or axial gauge transformations. The generator of this group is the corresponding smeared current, obeying Lundberg's commutation relations [14]. The implementing operators themselves give a multiplier representation of the corresponding current group.

We have said many times that the spatial rigid transformations are precisely the symmetries of the Hamiltonian. Indeed, it is an immediate consequence of the Wightman reconstruction theorem that if a rigid transformation is a symmetry of the Wightman functions, then it is spatial [15]. The converse of this is not so well known: if a symmetry is explicitly broken, say by a mass term, then it is never spatial, i.e. the symmetry is spontaneously broken as well. This follows from Haag's theorem, in the nice form given by Fabri and Picasso [16].

One can see, if masses are not zero, that an automorphism τ_T is spatial if and only if $T(\infty)$ is a symmetry: $T(\infty) \in \mathrm{G}_0$ (we have factored out one copy of G_0 by putting $T(-\infty)=1$). Thus we have verified the idea of Coleman [17], that topological states should be labelled by maps from spatial ∞ (here $\pm\infty$) to the unbroken part of the gauge group G_0. This result needs modifying if some masses are zero. It seems that $T(+\infty) \in \mathrm{G}_0$ then leads out of Fock space, but always gives a covariant state.

Proposition 1 seems to contradict the result, stated in [7], that a local gauge transformation is implemented for the massive case if and only if it is implemented for the massless case. But in fact these authors, and also Segal [18], postulate from the start that q or q_5 is $2\pi n$; they take space to be a torus.

In the fermion models 1, 2 and 3, the index is topologically unstable if we allow the transformations to run over the full set of real orthogonal transformations of the one-particle space regarded as a real space. Then the index is only conserved mod 2. We are not worried by this since these more general transformations are not complex linear and violate the gauge invariance: $\mathscr{A}_0$ is not mapped to itself.

The result 6(a) was announced by Bonnard and Streater [12]. In the verbatim report of a lecture on this model† it was stated that the automorphisms of this model lead to inequivalent representations of the observable algebra. This remains an open question.

† Published in 1975 *Trudy Mat. Inst. Steklov* **135** 83–8.

3. Proofs

Proposition 1. Recall that $\tau_{\alpha\beta}$ is spatial in π_m if and only if both $P_+^m T_{\alpha\beta} P_-^m$ and $P_-^m T_{\alpha\beta} P_+^m$ are Hilbert–Schmidt (HS) operators [19-20]. We work with the Fourier transformed operators $\hat{P}_\pm^m$ and $\hat{T}_{\alpha\beta}$ given by

$$(\hat{P}_\pm^m g)(k) = \frac{1}{2}\left(\mathbb{1} \pm \frac{k}{\omega(k)}\gamma_5 \pm \frac{m}{\omega(k)}\gamma_0\right) g(k), \tag{13}$$

where $\omega(k) = (k^2 + m^2)^{1/2}$ and

$$(\hat{T}_{\alpha\beta} g)(p) = \frac{1}{2\pi}\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px}(\mathrm{e}^{\mathrm{i}\alpha(x)}\cos\beta(x) + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha(x)}\sin\beta(x)\gamma_5)\int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}xq} g(q) \tag{14}$$

for $g \in L^2(\mathbb{R}, \mathbb{C}^2)$.

We write $L^2(\mathbb{R}, \mathbb{C}^2) = L^2(\mathbb{R}, \mathbb{C}) \oplus L^2(\mathbb{R}, \mathbb{C})$ and $\hat{P}_+^m \hat{T}_{\alpha\beta} \hat{P}_-^m$ (and similarly $\hat{P}_-^m \hat{T}_{\alpha\beta} \hat{P}_+^m$) is then a 2×2 matrix whose entries are operators on $L^2(\mathbb{R}, \mathbb{C})$. Thus

$$\hat{P}_+^m \hat{T}_{\alpha\beta} \hat{P}_-^m = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

is HS if and only if each entry is. Set $\eta_\pm(x) = \alpha(x) \pm \beta(x)$; S_{11} turns out to be

$$(S_{11}f)(p) = \frac{1}{8\pi}\int \mathrm{d}x\, \mathrm{d}q\, \exp[\mathrm{i}(q-p)x] \times \left(\frac{\exp(\mathrm{i}\eta_+(x))(\omega(p)\omega(q) + p\omega(q) - q\omega(p) - pq)}{\omega(p)\omega(q)} - \frac{\exp[\mathrm{i}\eta_-(x)]m^2}{\omega(p)\omega(q)}\right) f(q).$$

S_{11} can be split into the sum of two operators, $S_{11} = A + B$, with

$$(Af)(p) = (8\pi)^{-1}\int \mathrm{d}x\, \mathrm{d}q\, \exp[\mathrm{i}(q-p)x]\exp(\mathrm{i}\eta_+(x))\, W(p, q) f(q)$$

where

$$W(p, q) = \frac{\omega(p)\omega(q) + p\omega(q) - q\omega(p) - pq - m^2}{\omega(p)\omega(q)}$$

and

$$(Bf)(p) = (8\pi)^{-1}\int \mathrm{d}x\, \mathrm{d}q\, \exp[\mathrm{i}(q-p)x][\exp(\mathrm{i}\eta_+(x)) - \exp(\mathrm{i}\eta_-(x))]\frac{m^2}{\omega(p)\omega(q)} f(q).$$

To see that A is HS for all functions α, β considered, set $\gamma(x) = \exp(\mathrm{i}\eta_+(x))$. Observe that, in the sense of distributions, $\mathrm{i}p\hat{\gamma}(p) = \hat{\gamma}'(p)$ (here and in the sequel, $\hat{h}$ will be the Fourier transform of $h \in \mathcal{S}'(\mathbb{R})$ and $\hat{\gamma}'$ will denote the Fourier transform of $\gamma' = \mathrm{d}\gamma/\mathrm{d}x$). Since $\gamma' = \mathrm{i}\eta_+'(x)\exp(\mathrm{i}\eta_+(x))$ is in $\mathcal{S}$, $\hat{\gamma}'$ is too, so that $\mathrm{i}p\hat{\gamma}(p)$ is a distribution identifiable with a function of $\mathcal{S}(\mathbb{R})$, which we also call $\hat{\gamma}'$.

Thus

$$(Af)(p) = (8\pi)^{-1}\frac{\sqrt{2}\pi}{\mathrm{i}}\int \mathrm{d}q\, \hat{\gamma}'(p-q)\frac{W(p, q)}{p-q} f(q).$$

Now, $W(p,q)/(p-q)$ is in $C^\infty(\mathbb{R}^2)$ after it has been defined by continuity at $p=q$; and the following inequality holds:

$$\left|\frac{W(p,q)}{p-q}\right| = \left|\frac{\omega(q)-q}{\omega(q)}\,\frac{\omega(p)-\omega(q)+p-q}{\omega(p)(p-q)}\right| \leqslant 4\omega(p)^{-1}.$$

We thus estimate the Hilbert–Schmidt norm of A by

$$\int \mathrm{d}p\,\mathrm{d}q|A(p,q)|^2 = \frac{1}{32\pi}\iint \mathrm{d}p\,\mathrm{d}q\,|\hat{\gamma}'(p-q)|^2\left|\frac{W(p,q)}{p-q}\right|^2$$
$$\leqslant \frac{1}{2\pi}\iint \mathrm{d}p\,\mathrm{d}q\,|\hat{\gamma}'(p-q)|^2\frac{1}{\omega(p)^2}$$

and this is finite, since $\hat{\gamma}'\in\mathcal{S}(\mathbb{R})$.

As for the operator B, set $H(x)=\exp(\mathrm{i}\eta_+(x))-\exp(\mathrm{i}\eta_-(x))$

$$H(x)=H(x)-H(\infty)\theta(x)+H(\infty)\theta(x)=K(x)+H(\infty)\theta(x)$$

where $H(\infty)=\lim_{x\to\infty}H(x)$, $K(x)=H(x)-H(\infty)\theta(x)$ and $\theta(x)$ is the step function.

Now $K\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, so the kernel of B is divided up into the function $[(2\pi)^{1/2}/8\pi]m^2(\hat{K}(p-q))/(\omega(p)\omega(q))$ which is in $L^2(\mathbb{R}^2)$, and a part arising from $\theta(x)$, which is the crucial one. Recall that $\hat{\theta}(p)=z(\mathcal{P}(1/p)-\mathrm{i}\pi\delta(p))$ where z is a real numerical factor. Thus the following operator must be HS

$$(Rf)(p)=\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\omega(p)\omega(q)(p-q)}f(q)-\frac{\mathrm{i}\pi f(p)}{\omega(p)^2}.$$

Now let f be a real function; then we get

$$|(Rf)(p)|^2\geqslant\frac{\pi^2}{\omega(p)^4}|f(p)|^2$$

which shows that R is not HS, as a multiplication operator by a non-vanishing function cannot be such. We conclude that S_{11} is HS if and only if $H(\infty)=0$, which gives

$$0=\lim_{x\to\infty}[\exp(\mathrm{i}\eta_+(x))-\exp(\mathrm{i}\eta_-(x))]=2\mathrm{i}\lim_{x\to\infty}\exp(\mathrm{i}\alpha(x))\sin\beta(x)$$

i.e. if and only if $q=\lim_{x\to\infty}\beta(x)=n\pi$, $n=0,\pm1,\ldots$. This proves that if $q\neq n\pi$, $\hat{P}^m_\mp\hat{T}_{\alpha\beta}\hat{P}^m_-$ is not HS, so that $\tau_{\alpha\beta}$ is not spatial.

In order to prove that $\hat{P}^m_\mp\hat{T}_{\alpha\beta}\hat{P}^m_-$ is HS if $q=n\pi$ we must consider the other S_{ij}. The analysis of S_{22} is entirely similar to S_{11}, so we move on to S_{12}. We have

$$(S_{12}f)(p)=(8\pi)^{-1}\iint \mathrm{d}x\,\mathrm{d}q\exp[\mathrm{i}(q-p)x]$$
$$\times\left[-\frac{m}{\omega(q)}\frac{\omega(p)+p}{\omega(p)}[\exp(\mathrm{i}\eta_+(x))-\exp(\mathrm{i}\eta_-(x))]\right.$$
$$\left.+\frac{m}{\omega(p)}\frac{\omega(q)-\omega(p)+q-p}{\omega(q)}\exp(\mathrm{i}\eta_-(x))\right]f(q).$$

Again, the second term in square brackets gives a HS operator for any admissible choice of η_-, whereas the first, by the same argument developed for S_{11}, is HS if and only if $q=n\pi$. The analysis of S_{21} is identical, which shows that $\hat{P}^m_\mp\hat{T}_{\alpha\beta}\hat{P}^m_-$ is HS if and only if $q=n\pi$, $n=0,\pm1,\ldots$. The same result is proved for $\hat{P}^m_-\hat{T}_{\alpha\beta}\hat{P}^m_\mp$ by mimicking this proof, *mutatis mutandis*.

It is worth mentioning that the results for the massless case can easily be obtained: when $m=0$, $S_{12}=S_{21}=0$ and S_{11} is simply

$$(S_{11}f)(p)=\frac{1}{2\pi}\int \mathrm{d}x\,\mathrm{d}q\,\exp[\mathrm{i}(q-p)x]\exp(\mathrm{i}\eta_+(x))\theta(p)\theta(-q)f(q). \tag{15}$$

An easy calculation shows that the square of the HS norm is $\int_0^\infty \mathrm{d}p\int_{-\infty}^0 \mathrm{d}q|\widehat{\exp \mathrm{i}\eta_+}(p-q)|^2$ which is finite if and only if $\eta_+(x)\to 0$ as $x\to\infty$. For S_{22} we need $\eta_-(x)\to 0$ as $x\to\infty$, so that both $\alpha+\beta$ and $\alpha-\beta$ must have tails at ∞ equal to an integer multiple of 2π, which is the known result [3, 8].

Proof of proposition 2. We use the same criterion as for proposition 1, i.e. we check for which choices of the function θ the operators $\hat{P}_+\hat{T}_\theta\hat{P}_-$ and $\hat{P}_-\hat{T}_\theta\hat{P}_+$ (and the corresponding rigid operators) are HS. Here $\hat{P}_-$ is the Fourier transform of the projection $P_-=P_-^{m_1}\oplus P_-^{m_2}$ and $\hat{P}_+=1-\hat{P}_-$. Explicitly

$$(\hat{P}_\pm g)(k)=\begin{bmatrix}\frac{1}{2}\left(1\pm\frac{k}{\omega_1(k)}\gamma_5\pm\frac{m_1}{\omega_1(k)}\gamma_0\right) & 0\\ 0 & \frac{1}{2}\left(1\pm\frac{k}{\omega_2(k)}\gamma_5\pm\frac{m_2}{\omega_2(k)}\gamma_0\right)\end{bmatrix}\begin{bmatrix}g_{11}(k)\\ g_{12}(k)\\ g_{21}(k)\\ g_{22}(k)\end{bmatrix}.$$

To begin, $\hat{P}_+\hat{T}_\theta\hat{P}_-$, acting on the space $L^2(\mathbb{R},\mathbb{C}^4)$, may be written as a matrix of sixteen operators on $L^2(\mathbb{R},\mathbb{C})$ which we shall denote by $S_{ij,lm}$, where each of the pairs ij, lm runs through the values 11, 12, 21, 22. Similarly, $\hat{P}_-\hat{T}_\theta\hat{P}_+$ will consist of sixteen operators $S'_{ij,lm}$. If and only if all of the 32 operators $S_{ij,lm}$, $S'_{ij,lm}$ are HS will the automorphism τ_θ be spatial. Now it is useful to have a general sufficient condition for the convergence of an integral of the sort that one meets here. This condition is inserted here.

Convergence condition. Let W_1, W_2 be two measurable functions $\mathbb{R}\to\mathbb{R}^+$, bounded and satisfying both the following requirements.

(a) The restriction to $\mathbb{R}^+$ of either W_1 or W_2 belongs to $L^1(\mathbb{R}^+)$.

(b) The restriction to $\mathbb{R}^-$ of either W_1 or W_2 belongs to $L^1(\mathbb{R}^-)$.

Let G be a positive function in $L^1(\mathbb{R})$ such that $|x|G(x)$ also belongs to $L^1\mathbb{R}$.

Then $\iint_{\mathbb{R}^2}\mathrm{d}p\,\mathrm{d}q\,W_1(p)W_2(q)G(p-q)<\infty$. The proof of this condition proceeds quite simply by analysing separately what happens in the four regions bounded by the p and q axes. Set

$$\begin{aligned}I&=\iint_{\mathbb{R}^2}\mathrm{d}p\,\mathrm{d}q\,W_1(p)W_2(q)G(p-q)\\ &=\left(\int_{-\infty}^0\mathrm{d}p\int_{-\infty}^0\mathrm{d}q+\int_{-\infty}^0\mathrm{d}p\int_0^\infty\mathrm{d}q+\int_0^\infty\mathrm{d}p\int_{-\infty}^0\mathrm{d}q+\int_0^\infty\mathrm{d}p\int_0^\infty\mathrm{d}q\right)\\ &\quad\times(W_1(p)W_2(q)G(p-q)\\ &=I_1+I_2+I_3+I_4.\end{aligned}$$

Let $A>0$ be such that $W_1(p)\leqslant A$, $W_2(p)\leqslant A$, $\forall p\in\mathbb{R}$. Suppose also that the restriction of say W_1 belongs to $L^1(\mathbb{R}^+)$. Then we can estimate

$$\begin{aligned}I_4&\leqslant A\int_0^\infty\mathrm{d}p\int_0^\infty\mathrm{d}q\,W_1(p)G(p-q)=\int_0^\infty\mathrm{d}p\,W_1(p)\int_{-\infty}^p G(u)\,\mathrm{d}u\\ &\leqslant\int_0^\infty\mathrm{d}p\,W_1(p)\int_{-\infty}^\infty G(u)\,\mathrm{d}u<\infty,\end{aligned}$$

$$I_1=\int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}q\, W_1(-p)\,W_2(-q)G(q-p)$$
$$=\int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}q\, \tilde{W}_1(p)\,\tilde{W}_2(q)G(q-p)$$

where $\tilde{W}_i(p)=W_i(-p)$. Because of condition (b), $\tilde{W}_i$ obey (a) so the estimate for I_4 applies. For I_2 and I_3 we have

$$I_3=\int_0^\infty \mathrm{d}p \int_{-\infty}^0 \mathrm{d}q\, W_1(p)\,W_2(q)G(p-q)$$
$$\leqslant A^2\int_0^\infty \mathrm{d}p \int_{-\infty}^0 \mathrm{d}q\, G(p-q)$$
$$\leqslant A^2\int_0^\infty \mathrm{d}u\, uG(u)<\infty$$

and

$$I_2=\int_0^\infty \mathrm{d}p \int_{-\infty}^0 \mathrm{d}q\, \tilde{W}_1(p)\,\tilde{W}_2(q)\tilde{G}(p-q)<\infty$$

since $\tilde{W}_1$, $\tilde{W}_2$ are bounded and $\tilde{G}$ has the same integrability properties (a) and (b). If instead of W_1 belonging to $L^1(\mathbb{R}^+)$ we had W_1 belonging to $L^1(\mathbb{R}^-)$ when restricted to $\mathbb{R}^-$, the proof would run through symmetrically for I_1 and I_4, and identically for I_2 and I_3. This completes the proof of the convergence condition.

This condition, along with some knowledge acquired through the proof of proposition 1, will permit us to check the HS character of our 32 operators just by inspecting the kernels. This is lengthy but straightforward and we limit ourselves here to spelling out the details in a couple of cases, in part (a) of the proposition.

Proof of proposition 2(a).

$$(S_{11,11}g_{11})(p)=\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px}\cos\theta(x)\int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}qx}$$
$$\times\left[\left(1+\frac{p}{\omega_1(p)}\right)\left(1-\frac{q}{\omega_1(q)}\right)-\frac{m_1^2}{\omega_1(p)\omega_1(q)}\right]g_{11}(q)$$
$$=\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px}\cos\theta(x)$$
$$\times\int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}qx}\frac{\omega_1(p)\omega_1(q)+p\omega_1(q)-q\omega_1(p)-pq-m_1^2}{\omega_1(p)\omega_1(q)}g_{11}(q).$$

This operator does not mix the masses, and is HS for any admissible choice of θ, as is seen by observing that it is identical with the operator called A in the proof of proposition 1, with the function $\exp(\mathrm{i}\eta_+(x))$ replaced here by $\cos\theta(x)$.

A more crucial, mass-mixing, operator is

$$(S_{11,21}g_{21})(p)=\int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px}\sin\theta(x)\int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}qx}\left(\frac{(\omega_1(p)+p)(\omega_2(q)-q)-m_1m_2}{\omega_1(p)\omega_2(q)}\right)g_{21}(q).$$

The function in brackets does not vanish when $p=q$, as $m_1 \neq m_2$. Then if $Q = \lim_{x\to\infty} \theta(x) = n\pi$, so that $\sin\theta(x) \in \mathscr{S}(\mathbb{R})$, put $g(p) = \sqrt{2\pi}(\sin\theta)\hat{}(p)$; the integral of the squared kernel is then

$$\iint \mathrm{d}p\,\mathrm{d}q \left(\frac{(\omega_1(p)+p)(\omega_2(q)-q)-m_1m_2}{\omega_1(p)\omega_2(q)}\right)^2 |g(p-q)|^2$$

and it satisfies the assumptions of the convergence condition as $W_1 = [(\omega_1(p)+p)/\omega_1(p)]^2$, $W_2 = [(\omega_2(q)-q)/\omega_2(q)]^2$ are both positive and bounded, and $W_1 \in L^1(\mathbb{R}^-)$ on $\mathbb{R}^-$, while $W_2 \in L^1(\mathbb{R}^+)$ on $\mathbb{R}^+$. Thus $S_{11,21}$ is HS. All the other operators are dealt with in an entirely similar way. One feature that is obviously everywhere present is that the operators containing $\cos\theta(x)$ are non-mass-mixing, and are HS as in proposition 1, even when $Q \neq \pi$ say. The mass-mixing terms contain $\sin\theta(x)$; here implementability is helped by the zero of $\sin\theta(x)$ when $Q = n\pi$. The same phenomenon occurs in the boson proof, proposition 6.

The operators $S'_{ij;lm}$ are dealt with by changing the signs of ω_1 and ω_2. This does not upset any convergence properties.

To complete proposition 2(a) it remains to prove that for $Q \neq n\pi$ the automorphism is not implemented. To this end it is enough to find one of the S_{ijlm}, S'_{ijlm} that is not HS.

Consider

$$(S_{11,22}g_{22})(p) = \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px} \sin\theta(x) \int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}xq} \times\left[\frac{m_1}{\omega_1(p)}\left(1+\frac{q}{\omega_2(q)}\right) - \frac{m_2}{\omega_2(q)}\left(1+\frac{p}{\omega_1(p)}\right)\right] g_{22}(q)$$

and split $\sin\theta(x)$ as $\sin\theta(x) = K(x) + q\Theta(x)$ as in the proof of proposition 1, with $K \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $q = \lim_{x\to\infty} \sin\theta(x) \neq 0$. Now the part of the kernel containing K is square-integrable, whereas the part coming from $q\Theta(x)$ is not, as the pole of $\hat{\Theta}(p)$ at $p=0$ is not cancelled by the vanishing of

$$\left.\frac{m_1}{\omega_1(p)}\left(1+\frac{q}{\omega_2(q)}\right) - \frac{m_2}{\omega_2(q)}\left(1+\frac{p}{\omega_1(p)}\right)\right|_{p=q}$$

which is not zero if $m_1 \neq m_2$.

This proves part (a).

Proofs of propositions 2(b) and 2(c). We noticed in the proof of part (a) that the non-implementability when $Q \neq n\pi$ was caused by the factor $\hat{\Theta}(p)$ which is not cancelled if $m_1 \neq m_2$, but is if $m_1 = m_2$. This, together with techniques of proposition 1, is enough to prove 2(b).

For 2(c), where one of the masses is zero, implementability is obtained for $Q = 2n\pi$ only, by the same techniques.

Proof of proposition 3. (a) When the masses are zero, the positive-energy projection is

$$P_+ = \bigoplus_{i=1}^{n} \begin{pmatrix} \Theta(p) & 0 \\ 0 & \Theta(-p) \end{pmatrix}$$

and the general T_{UV} element has the form

$$T_{UV} = W = \begin{pmatrix} \alpha_1 & 0 & \cdots & \beta_1 & 0 & \cdots \\ 0 & \alpha_2 & & 0 & \beta_2 & \cdots \\ \vdots & & \ddots & & & \end{pmatrix}$$

where $\alpha, \beta \ldots$ denote smooth complex-valued functions on $\cdot\mathbb{R}$ and also, in the sequel, the multiplication operators by α and β. The typical elements of $P_+ WP_-$ are

$$\Theta(p)\alpha_1\Theta(-p) \qquad \Theta(p)\beta_1\Theta(-p), \qquad \Theta(-p)\alpha_2\Theta(p) \qquad \text{and} \qquad \Theta(-p)\beta_2\Theta(p).$$

Just as in the case $n=1$, we are led to integrals of the type (15), with α_j, β_j replacing $\exp(i\eta_+(x))$ there. These are then HS if and only if $\alpha_j(+\infty)=\alpha_j(-\infty)$, $\beta_j(+\infty)=\beta_j(-\infty)$ which says $W(\infty)=1$.

(b) In the massive case, the projection onto positive-energy states is

$$P_+ = \bigoplus_{i=1}^{n} P_+^{m_i} = \bigoplus_{i=1}^{n} \frac{1}{2}\left(1+\frac{p}{(p^2+m_i^2)^{1/2}}\gamma_5+\frac{m_i}{(p^2+m_i^2)^{1/2}}\gamma_0\right) \tag{16}$$

acting on the direct sum $\bigoplus_{i=1}^{n} L^2(\mathbb{R},\mathbb{C}^2)$. Then writing ω_j for the pseudo-differential operator $(m_j^2+p^2)^{1/2}$:

$$P_+WP_- = \begin{bmatrix} 1+p\omega_1^{-1} & \dfrac{m_1}{\omega_1} & & \\ \dfrac{m_1}{\omega_1} & 1-p\omega_1^{-1} & \ldots 0 \ldots & \\ \vdots & \ddots & 1+\dfrac{p}{\omega_n} & \dfrac{m_n}{\omega_n} \\ 0 & & \dfrac{m_n}{\omega_n} & 1-\dfrac{p}{\omega_n} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & \beta_1 & 0 & \\ 0 & \alpha_2 & \ldots & 0 & \beta_2 & \ldots \\ \vdots & \ddots & & & \end{bmatrix} \begin{bmatrix} 1-\dfrac{p}{\omega_1} & \dfrac{-m_1}{\omega_1} & & \\ \dfrac{-m_1}{\omega_1} & 1+\dfrac{p}{\omega_1} & & \\ & & \ddots & \\ & & 1-\dfrac{p}{\omega_n} & \dfrac{-m_n}{\omega_n} \\ & & \dfrac{-m_n}{\omega_n} & 1+\dfrac{p_n}{\omega_n} \end{bmatrix}$$

There are eight typical terms, each of which must be HS for τ_W to be spatial:

$$\begin{aligned}
&(1+p\omega_1^{-1})\alpha_1(1-p\omega_1^{-1})-m_1^2\omega_1^{-1}\alpha_2\omega_1^{-1},\\
&-(1+p\omega_1^{-1})\alpha_1 m_1\omega_1^{-1}+m_1\omega_1^{-1}\alpha_2(1+p\omega_1^{-1}),\\
&(1+p\omega_1^{-1})\beta_1(1-p\omega_2^{-1})-m_1m_2\omega_1^{-1}\beta_2\omega_2^{-1},\\
&-(1+p\omega_1^{-1})\beta_1 m_2\omega_2^{-1}+m_1\omega_1^{-1}\beta_2(1+p\omega_2^{-1}),\\
&m_1\omega_1^{-1}\alpha_1(1-p\omega_1^{-1})-m_1(1-p\omega_1^{-1})\alpha_2\omega_1^{-1},\\
&-m_1^2\omega_1^{-1}\alpha_1\omega_1^{-1}+(1-p\omega_1^{-1})\alpha_2(1+p\omega_1^{-1}),\\
&m_1\omega_1^{-1}\beta_1(1-p\omega_2^{-1})+(1-p\omega_1^{-1})\beta_2(-m_2\omega_2^{-1}),\\
&-m_1m_2\omega_1^{-1}\beta_1\omega_2^{-1}+(1-p\omega_1^{-1})\beta_2(1+p\omega_2^{-1}).
\end{aligned}$$

Take the first term; write $-\omega_1^{-1}\alpha_2\omega_1^{-1}=\omega_1^{-1}(\alpha_1-\alpha_2)\omega_1^{-1}-\omega_1^{-1}\alpha_1\omega_1^{-1}$. The kernel of $\omega_1^{-1}(\alpha_1-\alpha_2)\omega_1^{-1}$ is

$$(p^2+m_1^2)^{-1/2}[\tilde{\alpha}_1(p-q)-\tilde{\alpha}_2(p-q)](q^2+m_1^2)^{-1/2}$$

which is HS if and only if the functions α_1 and α_2 have the same value at ∞, so that $\alpha_1-\alpha_2 \in \mathcal{D}$. This is therefore a necessary condition, as cancellation with other terms is not possible. So with this condition we can replace α_2 by $\alpha_1=\alpha$ and the first term becomes

$$\begin{aligned}
&\alpha - p\omega_1^{-1}\alpha p\omega_1^{-1}-m_1\omega_1^{-1}\alpha m_1\omega_1^{-1}+p\omega_1^{-1}\alpha-\alpha p\omega_1^{-1}\\
&\quad=\underbrace{\alpha-p^2\omega_1^{-2}\alpha-m_1^2\omega_1^{-2}\alpha}_{\substack{\| \\ 0}}+p\omega_1^{-1}[p\omega_1^{-1},\alpha]+m_1\omega_1^{-1}[m_1\omega_1^{-1},\alpha]+[p\omega_1^{-1},\alpha].
\end{aligned}$$

Since $p\omega_1^{-1}$ and $m_1\omega_1^{-1}$ are bounded, it is enough to show that $[p\omega_1^{-1}, \alpha]$ and $[m_1\omega_1^{-1}, \alpha]$ are HS. The kernel of the first is, putting $m = m_1$:

$$\frac{p}{(p^2+m^2)^{1/2}}\tilde{\alpha}(p-q)-\tilde{\alpha}(p-q)\frac{q}{(q^2+m^2)^{1/2}}=[(p-q)\tilde{\alpha}(p-q)]\left(\frac{p\omega(p)^{-1}-q\omega(q)^{-1}}{p-q}\right).$$

As in proposition 1, $(p-q)\tilde{\alpha}(p-q)$ is in $\mathcal{S}(\mathbb{R})$ in $p-q$, and the product is therefore HS. The only condition is $\alpha' \in \mathcal{S}(\mathbb{R})$, true as W is a constant outside a compact set. Similarly $[m\omega_1^{-1}, \alpha]$ is HS. This deals with the first term.

This deals with the second term too, since $\omega_1^{-1}(\alpha_2-\alpha_1)(1+p\omega_1^{-1})$ is HS if $\alpha_2(\infty) = \alpha_1(\infty)$. So again we may replace α_1 and α_2 by α say, when the second term becomes $m_1\omega_1^{-1}[\alpha, p]\omega_1^{-1} + m_1[\omega_1^{-1}, \alpha]$ which is HS.

A harder term is the third, involving β and two masses. If $m_1 = m_2$, then this is similar to the first term: it is HS if and only if $\beta_1 = \beta_2$ at ∞; but this is then the condition that, at ∞, W should be one of the exact rigid symmetries of the theory.

If $m_1 \neq m_2$ we must consider

$$(\beta_1-\beta_1 p\omega_1^{-1}+p\omega_1^{-1}\beta_1-p\omega_1^{-1}\beta_1 p\omega_1^{-1}-m_1^2\omega_1^{-1}\beta_1\omega_1^{-1})+\beta_1 p\omega_1^{-1}-\beta_1 p\omega_2^{-1}$$
$$+p\omega_1^{-1}\beta_1 p\omega_1^{-1}-p\omega_1^{-1}\beta_1 p\omega_2^{-1}+m_1^2\omega_1^{-1}\beta_1\omega_1^{-1}-m_1m_2\omega_1^{-1}\beta_2\omega_2^{-1}.$$

We have arranged to cancel β_1, which is clearly not HS, under the general conditions of theorem 3(b), i.e. $\beta_1 \to$ constant at ∞. Indeed, the term ( ) is treated as for the first term.

If β_1 or β_2 is non-zero at ∞, at least one of the remaining terms has a kernel with a $1/(p-q)$ singularity at $p=q$; the kernel is an analytic function of p and q, so to be HS this singularity must cancel identically for all values of the other variable, say $p+q$. This is not possible; hence β_1 and β_2 vanish at ∞. Conversely, if β_1 and β_2 do vanish at ∞, the remaining terms can be paired

$$\beta_1 p(\omega_1^{-1}-\omega_2^{-1})+p\omega_1^{-1}\beta_1 p(\omega_1^{-1}-\omega_2^{-1})+m_1^2\omega_1^{-1}\beta_1\omega_1^{-1}-m_1m_2\omega_1^{-1}\beta_2\omega_2^{-1}$$

and each of these is HS as $\beta_1, \beta_2 \in \mathcal{S}(\mathbb{R})$.

The remaining five terms are then very similarly shown to be HS.

We conclude that τ_{UV} is spatial if and only if $T_{UV}(\infty)$ is an exact rigid symmetry. This proves proposition 3.

When $n=1$, and α_1, α_2 converge to 1 at ∞, then the implementability of τ_U was proved in [7]. This method does not work, however, when $\alpha_1, \alpha_2 \to -1$ at $+\infty$, which is spatial and actually gives the 'one-soliton' state.

Proof of proposition 4. We now show that if all masses are positive then the gauge solitons are topologically unstable, in that the index of P_+WP_+, where $W = T_{UU}$ contains no axial part, is zero.

Proof. It is easy to show that P_+^m is a norm continuous function of m when $m > 0$. The same goes for $P_+ = \bigoplus_i P_+^{m_i}$ as a function of $m_1, \ldots, m_n$ in the region $m_k > 0$. Then the estimate

$$\|P_+WP_+ - Q_+WQ_+\| \leq 2\|P_+ - Q_+\|$$

shows that P_+WP_+ is a continuous Fredholm family as discussed by Atiyah [21]. Therefore for fixed W the Fredholm index is independent of $m_1, \ldots, m_n$ as long as these are greater than zero. Let us then move them to become all equal.

Let us now move W through the set of C^∞ maps $X: \mathbb{R} \to M(2n)$, constant outside some compact set. The operator norm of X on $L^2(\mathbb{R}, \mathbb{C}^{2n})$ obeys

$$\|X\| \leq |||X||| = (2n)^2 \sup_x \sup_{i,j} |X_{ij}(x)|.$$

Then the inequality

$$\|P_+WP_+ - P_+XP_+\| \leq \|W - X\| \leq |||W - X|||$$

shows that the set of Fredholm operators P_+WP_+ is again a continuous family, and so the Fredholm index is constant on any $|||\ |||$ connected set of spatial gauge transformations. We note that from proposition 3(b), since the masses are now all equal, any value of $W(\infty)$ ensures that P_+WP_+ is Fredholm. We can now move W continuously in $|||\ |||$-norm to an operator near the identity: e.g. let

$$W^\lambda = \begin{cases} W^\lambda(x) = W(x), & x \leq b - \lambda \\ W^\lambda(x) = W(b-\lambda), & x > b - \lambda + \varepsilon \\ |||W^\lambda(x) - W(b-\lambda)||| < \varepsilon, & \text{if } b - \lambda < x < b - \lambda + \varepsilon \end{cases}$$

for ε small (this ε is needed to ensure $W \in C^\infty$ and is not just continuous). This can be done so that W^λ is $|||\ |||$-continuous in λ.

Then as λ moves from 0 to $b - a$ we move continuously to a matrix, the identity outside $[a, a + \varepsilon]$, for which $P_+W^bP_+$ has zero index for some ε.

Proof of proposition 5. If the masses are all zero, the energy projections take the form

$$P_\pm = \begin{pmatrix} \Theta(\pm p) & 0 \\ 0 & \Theta(\mp p) \end{pmatrix}$$

acting on each direct summand $L^2(\mathbb{R}, \mathbb{C}^2)_i$, $i = 1, 2, \ldots, n$. We can then follow Carey *et al* [7] replacing $\mathbb{C}^2$ by $\mathbb{C}^{2n}$. The separation into top and bottom components of each $\psi_1, \ldots, \psi_n$ is a relativistically invariant notion. Consider then the n top components and subject them to a unitary transformation $U(x)$, with $U(-\infty) = 1_n = U(\infty)$, needed for τ_{U1} to be spatial (proposition 3(a)). The bottom components we treat separately, acting by the unitary operator $V(x)$. If $V = U$ we have a gauge transformation, and if $V = U^*$, an axial gauge transformation. Because $U(\pm\infty) = 1$ we can regard U as a function on T, the compactified $\mathbb{R}$, by the mapping $t = 2\tan^{-1}x$, $x \in \mathbb{R}$, $t \in [-\pi, \pi)$. This induces a unitary map S: $L^2(T, \mathbb{C}^n) \to L^2(\mathbb{R}, \mathbb{C}^n)$ defined by

$$(Sf)_k(x) = f_k(2\tan^{-1}x)/(x - i).$$

The projections $E_\pm = S^{-1}P_\pm S$ act on $L^2(T, \mathbb{C}^n)$ and project it to the Hardy space of functions on T^n with $\pm$'ve multiple Fourier coefficients. As usual $S^{-1}US = \tilde{W}$ takes the form

$$\begin{pmatrix} E_+\tilde{W}E_+ & E_+\tilde{W}E_- \\ E_-\tilde{W}E_+ & E_-\tilde{W}E_- \end{pmatrix}.$$

Namely, $E_+\tilde{W}E_-$ is HS and $E_+\tilde{W}E_+$ is a Toeplitz operator of the form discussed by Douglas [22]. As remarked in [7], this shows the implementability of these gauge transformations with no need to compute the kernels explicitly.

We now apply the result [22] or [21, 23], showing that the index of $E_+\tilde{W}E_+$ is minus the winding number of det U about the origin. As in [22], this index is unchanged if we move U through unitary matrices, so that all the wind is in, say, ψ_1. We can then use [7] to identify the index as the net number of right-going particles (antiparticles counted negatively). Moving U back to a general element with the same index needs an implementable automorphism creating no charge (Labonté [9]). Hence the winding number of det U for a general U is minus the charge created.

Similarly, the bottom component V is related to the net left-going charge. To be consistent with parity, charge is this time equal to the winding number (i.e. minus the winding number from right to left, as the particles are now left-going).

Proof of proposition 6. For the boson model 4 we follow mainly [12]. The one-particle space is $\bigoplus_{i=1}^4 L^2(\mathbb{R}, \mathbb{R})_i$, subjected to real symplectic transformations, induced by

$$\Theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

of the form

$$T = \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix} \begin{pmatrix} \gamma & \\ & \gamma^{-1} \end{pmatrix} \qquad \text{where } \gamma = \begin{pmatrix} \gamma_1 & \\ & \gamma_2 \end{pmatrix}$$

$\gamma_i = (p^2 + m_i^2)^{1/4}$, and T acts on $(\phi_1, \phi_2, \pi_1, \pi_2)$. The Shale criterion is that, to be spatial, $1 - T^t T$ must be HS. We compute

$$T^t T = \begin{pmatrix} \gamma & \\ & \gamma^{-1} \end{pmatrix} \begin{pmatrix} \Theta^{-1} & \\ & \Theta^{-1} \end{pmatrix} \begin{pmatrix} \gamma^{-2} & \\ & \gamma^2 \end{pmatrix} \begin{pmatrix} \Theta & \\ & \Theta \end{pmatrix} \begin{pmatrix} \gamma & \\ & \gamma^{-1} \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} \gamma_1 c\gamma_1^{-2} c\gamma_1 + \gamma_1 s\gamma_2^{-2} s\gamma_1 & \gamma_1 c\gamma_1^{-2} s\gamma_1 - \gamma_1 s\gamma_2^{-2} c\gamma_1 & \\ \gamma_2 s\gamma_1^{-2} c\gamma_2 - \gamma_2 c\gamma_2^{-2} s\gamma_2 & \gamma_2 s\gamma_1^{-2} s\gamma_2 + \gamma_2 c\gamma_2^{-2} c\gamma_2 & 0 \\ \hline \multicolumn{2}{c|}{0} & \gamma \leftrightarrow \gamma^{-1} \end{array}\right)$$

$c = \cos\theta(x)$, $s = \sin\theta(x)$.

The diagonal terms will be shown to be of the form $1 + \text{HS}$ using $c^2 + s^2 = 1$, if and only if $\theta(\infty) = n\pi$. We first push the terms involving the mass difference onto the terms involving s, thus:

$$s\gamma_2^{-2} s = s\gamma_1^{-2} s + s(\gamma_2^{-2} - \gamma_1^{-2})s \qquad \text{for } 1, 1$$
$$s\gamma_1^{-2} s = s\gamma_2^{-2} s + s(\gamma_1^{-2} - \gamma_2^{-2})s \qquad \text{for } 2, 2$$
$$s\gamma_2^2 s = s\gamma_1^2 s + s(\gamma_2^2 - \gamma_1^2)s \qquad \text{for } 3, 3$$
$$s\gamma_1^2 s = s\gamma_2^2 s + s(\gamma_1^2 - \gamma_2^2)s \qquad \text{for } 4, 4.$$

Then the 1, 1 term is

$$\gamma_1 c\gamma_1^{-2} c\gamma_1 + \gamma_1 s\gamma_2^{-2} s\gamma_1$$
$$= \gamma_1[c, \gamma_1^{-2}]c\gamma_1 + \gamma_1\gamma_1^{-2}c^2\gamma_1 + \gamma_1[s, \gamma_1^{-2}]s\gamma_1 + \gamma_1\gamma_1^{-2}s^2\gamma_1 + \gamma_1 s(\gamma_2^{-2} - \gamma_1^{-2})s\gamma_1.$$

Since $\gamma_1\gamma_1^{-2}c^2\gamma_1 + \gamma_1\gamma_1^{-2}s^2\gamma_1 = 1$, it remains to show that every remaining term is HS if $Q = n\pi$ when $m_1 \neq m_2$, and for any value of this limit if $m_1 = m_2$.

To analyse the first term we note that whereas $c\gamma_1$ is not bounded, $c\gamma_1 = [c, \gamma_1] + \gamma_1 c$, where $[c, \gamma_1]$ is bounded; its kernel is

$$[c, \gamma_1](p, q) = \frac{(q^2 + m^2)^{1/4} - (p^2 + m^2)^{1/4}}{p - q}(p - q)\hat{c}(p - q).$$

Change variables to $t=\frac{1}{2}(p-q)$, $k=\frac{1}{2}(p+q)$. Then the kernel of $[c, \gamma_1]$ is

$$t\hat{c}(2t)\left\{\frac{|k|^{1/2}}{t}\left[\left(1+\frac{2t}{k}+\frac{t^2+m^2}{k^2}\right)^{1/4}-\left(1-\frac{2t}{k}+\frac{t^2+m^2}{k^2}\right)^{1/4}\right]\right\}.$$

Since $t\hat{c}(2t)\in\mathscr{S}(\mathbb{R})$, and the $\{\ \}$-factor lies in L^3 as a function of k, this operator is compact. Similarly $[c, \gamma_2]$, $[s, \gamma_1]$ and $[s, \gamma_2]$ are compact. Thus we may write

$$\gamma_1[c, \gamma_1^{-2}]c\gamma_1=\gamma_1[c, \gamma_1^{-2}]\gamma_1 c+\gamma_1[c, \gamma_1^{-2}][c, \gamma_1]$$
$$\gamma_1[s, \gamma_1^{-2}]s\gamma_1=\gamma_1[s, \gamma_1^{-2}]\gamma_1 s+\gamma_1[s, \gamma_1^{-2}][s, \gamma_1].$$

Since c, s, $\gamma_1^{-1}[c, \gamma_1]$ and $\gamma_1^{-1}[s, \gamma_1]$ are bounded, it is enough to show that $\gamma_1[c, \gamma_1^{-2}]\gamma_1$ and $\gamma_1[s, \gamma_1^{-2}]\gamma_1$ are HS to deal with these terms. In the variables t, k the kernel of $\gamma_1[c, \gamma_1^{-2}]\gamma_1$ is

$$t^{-1}[((t+k)^2+m_1^2)^{1/4}((t-k)^2+m_1^2)^{-1/4}-((t+k)^2+m_1^2)^{-1/4}((t-k)^2+m_1^2)^{1/4}]t\hat{c}(2t)$$
$$=(1/2k+\mathrm{O}(1/k^2))t\hat{c}(2t) \qquad \text{as } k\to\infty.$$

This lies in $L^2(\mathbb{R}^2)$ since $t\hat{c}\in\mathscr{S}$. Similarly $\gamma_1[s, \gamma_1^{-2}]\gamma_1$ is HS.

The mass-difference term can be written

$$\gamma_1 s(\gamma_2^{-2}-\gamma_1^{-2})\gamma_1 s+\gamma_1 s(\gamma_2^{-2}-\gamma_1^{-2})\gamma_1\gamma_1^{-1}[s, \gamma_1].$$

Since s and $\gamma_1^{-1}[s, \gamma_1]$ are bounded, it is enough to show that $\gamma_1 s(\gamma_2^{-2}-\gamma_1^{-2})\gamma_1$ is HS. Its kernel is

$$(p^2+m_1^2)^{1/4}\hat{s}(p-q)[(q^2+m_2^2)^{-1/2}-(q^2+m_1^2)^{-1/2}](q^2+m_1^2)^{1/4}$$

which is $\hat{s}(t)(m_2^2-m_1^2)\mathrm{O}(1/k^2)$ as $k\to\infty$, which is HS.

Notice that we do not get a factor t, so we need $\hat{s}(t)\in\mathscr{S}$ and cannot use $t\hat{s}(t)\in\mathscr{S}$.

So far we have shown that the $(1, 1)$ term is of the form $1+\mathrm{HS}$ if $Q=n\pi$. If, however, $Q\neq n\pi$, then all terms are HS except $\gamma_1 s(\gamma_2^{-2}-\gamma_1^{-2})s\gamma_1$, whose kernel

$$\int \mathrm{d}k\,(p^2+m_1^2)^{1/4}\hat{s}(p-k)[\gamma_2^{-2}(k)-\gamma_1^{-2}(k)]\hat{s}(k-q)(q^2+m_1^2)^{1/4}$$

has a 'pinch' singularity at $p=q$, coming from the factors $\int \mathrm{d}k/(p-k-\mathrm{i}\varepsilon)(k-q+\mathrm{i}\varepsilon)$. This is not removable, and so this term is not HS, unless $m_1=m_2$, when it is zero. For the $(1, 2)$ off-diagonal term, we write

$$\gamma_1 c\gamma_1^{-2}s\gamma_1-\gamma_1 s\gamma_2^{-2}c\gamma_1=\gamma_1 c[\gamma_1^{-2}, s]\gamma_1-\gamma_1 s[\gamma_2^{-2}, c]\gamma_1+\gamma_1 sc(\gamma_1^{-1}-\gamma_2^{-2}\gamma_1).$$

The first two terms have been shown to be HS. If sc has compact support, then $sc\in\mathscr{S}$ and the last term has kernel $\mathrm{O}(k^{1/2})\widehat{sc}(2t)\mathrm{O}(k^{-1/2})(m_2^2-m_1^2)/k^2$ as $k\to\infty$, which is in L^2. But if $sc\neq 0$ at ∞ we get a pole $1/t$ which is not cancelled unless $m_1=m_2$ (when this term vanishes).

Similarly all the four terms written $(2, 1)$, $(2, 2)$, are HS or $1+\mathrm{HS}$ if $Q=n\pi$, otherwise not.

For the terms $(3, 3)$, $(3, 4)$ and $(4, 4)$ an entirely similar calculation can be done: one uses that $\gamma_1^{-1}[c, \gamma_1^2]\gamma_1^{-1}\in\mathscr{S}$ and $\gamma_1[c, \gamma_1^{-1}]$ is bounded, etc.

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